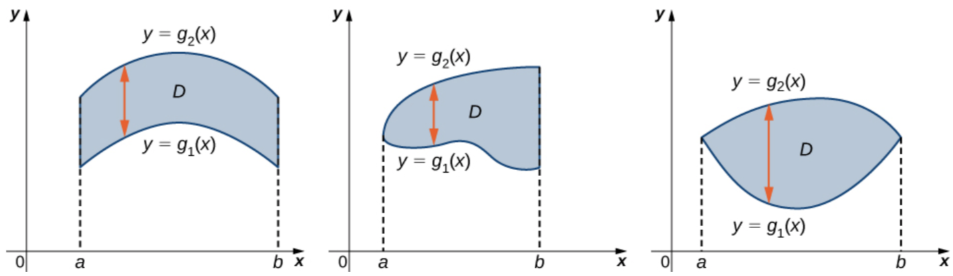
SECTION 16.2: DOUBLE INTEGRALS OVER GENERAL REGIONS

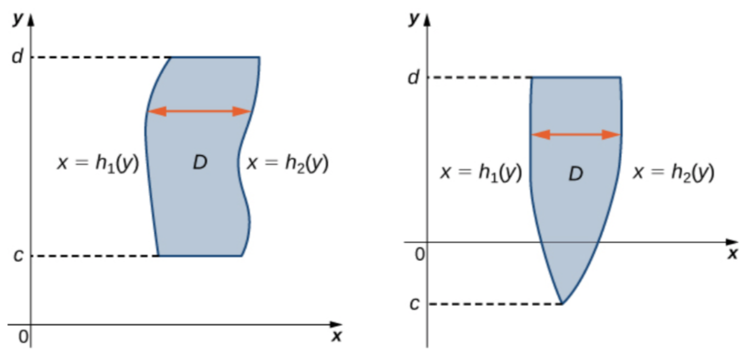
Suppose we have a non-rectangular region in the plane.

- A 'Type I' Region:



$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_{\text{left constant}}^{\text{right constant}} \int_{\text{bottom curve}}^{\text{top curve}} f(x,y)\,dy\,dx$$

- A 'Type II' Region:



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_{\text{bottom constant}}^{\text{top constant}} \int_{\text{left curve}}^{\text{right curve}} f(x,y)\,dx\,dy$$

EXAMPLE 1: Sketch the region over which $\int_0^1 \int_0^{x^2} xe^{2y}\,dy\,dx$ is being evaluated and evaluate the integral.

Ans: $\int_0^1 \int_0^{x^2} xe^{2y}\,dy\,dx = \dfrac{e^3 - 3}{8}$

EXAMPLE 2: Consider the integral: $\displaystyle\int_0^1 \int_{2x}^2 \sin\left(y^2\right)\, dy\, dx$

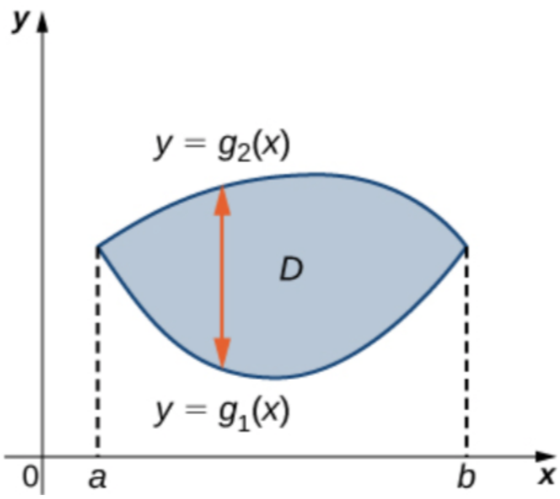
1. What goes wrong when trying to evaluate this integral?

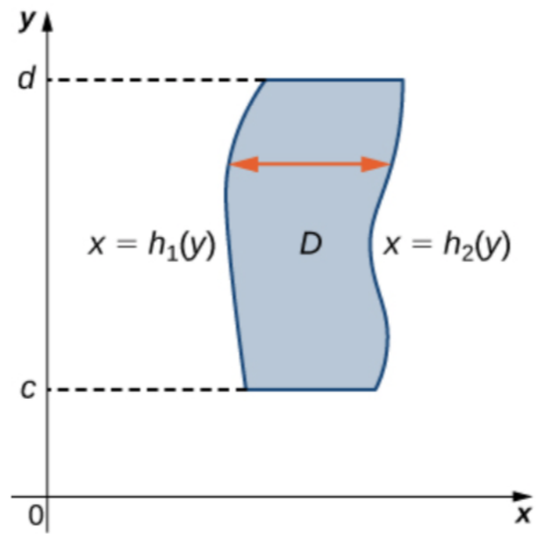
2. Sketch the region over which this integral is being evaluated.

3. Switch the order of integration and evaluate the equivalent integral.

Ans: $\displaystyle\int_0^1 \int_{2x}^2 \sin\left(y^2\right)\, dy\, dx = \int_0^2 \int_0^{y/2} \sin\left(y^2\right)\, dx\, dy = \frac{1-\cos(4)}{4}$

AREA USING A DOUBLE INTEGRAL:





$$\text{Area} = \int_a^b [g_2(x) - g_1(x)]\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} 1\,dy\,dx \qquad \text{Area} = \int_c^d [h_2(y) - h_1(y)]\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} 1\,dx\,dy$$

In general,

$$\text{Area of a Region } R = \iint_R 1\,dA$$

EXAMPLE 3: Let R be the region bounded by $y = \sqrt{x}$, $y = 6 - x$ and the x-axis.

1. Set up iterated integral(s) with integration order $dy\,dx$ which would compute the area of R.

$$\text{Ans: } \int_0^4 \int_0^{\sqrt{x}} 1\,dy\,dx + \int_4^6 \int_0^{6-x} 1\,dy\,dx = \frac{22}{3}\text{ units}^2$$

2. Set up iterated integral(s) with integration order $dx\,dy$ which would compute the area of R.

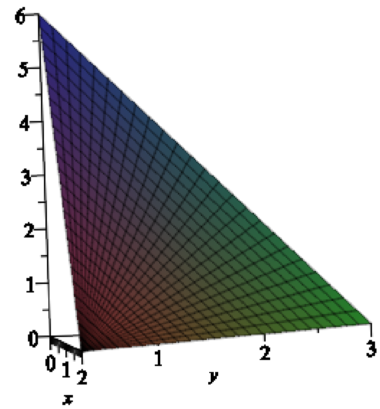
$$\text{Ans: } \int_0^2 \int_{y^2}^{6-y} 1\,dy\,dx = \frac{22}{3}\text{ units}^2$$

VOLUME USING A DOUBLE INTEGRAL:

$$\textbf{BIG IDEA: } \text{Volume} = \iint_{\text{base}} \text{height}\, dA$$

- Choose a coordinate plane to act as the 'base.'
  - If you choose the xy-plane, the height is determined by $z = f(x, y)$.
  - If you choose the yz-plane, the height is determined by $x = f(y, z)$.
  - If you choose the xz-plane, the height is determined by $y = f(x, z)$.
- Project the solid into the coordinate plane to draw a detailed sketch of the base. Algebra helps here!
- Using a detailed drawing of the base, choose a convenient order of integration.

EXAMPLE 4: Consider the solid bounded by the plane $2x + y + 3z = 6$ in the first octant.



1. Set up an iterated integral with order $dy\,dx$ which would compute the volume of the solid.

Ans: $\displaystyle\int_0^3 \int_0^{-2x+6} \left(-\frac{2}{3}x - \frac{1}{3}y + 2\right)\,dy\,dx = 6\,\text{units}^3$

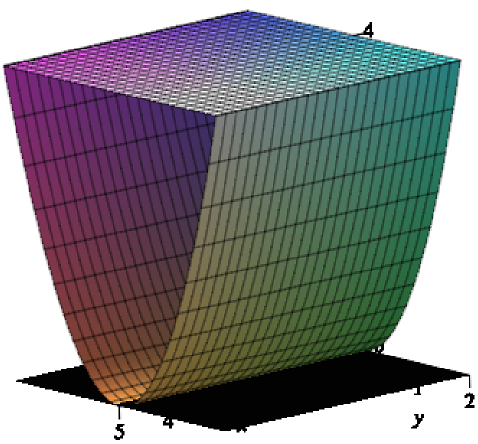
2. Set up an iterated integral with order $dy\,dz$ which would compute the volume of the solid.

Ans: $\displaystyle\int_0^2 \int_0^{-3z+6} \left(-\frac{1}{2}y - \frac{3}{2}z + 3\right)\,dy\,dz = 6\,\text{units}^3$

3. Set up an iterated integral with order $dx\,dz$ which would compute the volume of the solid.

Ans: $\displaystyle\int_0^2 \int_0^{-\frac{3}{2}z+3} (-2x - 3z + 6)\,dx\,dz = 6\,\text{units}^3$

EXAMPLE 5: Consider the solid bounded by $z = y^2$ and the planes $z = 4$, $x = 0$ and $x = 5$.



1. Set up an iterated integral with order $dy\,dx$ which would compute the volume of the solid.

Ans: $\displaystyle\int_0^5 \int_{-2}^2 \left(4 - y^2\right)\,dy\,dx = \frac{160}{3}$ units^3

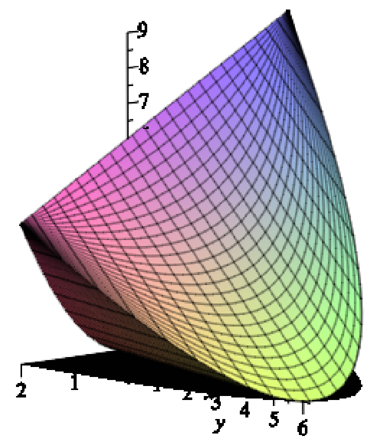
2. Set up an iterated integral with order $dz\,dy$ which would compute the volume of the solid.

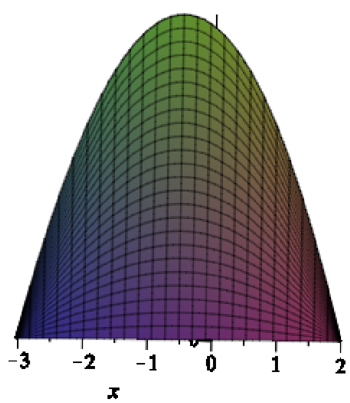
Ans: $\displaystyle\int_{-2}^2 \int_{y^2}^4 5\,dz\,dy = \frac{160}{3}$ units^3

3. Set up an iterated integral with order $dz\,dx$ which would compute the volume of the solid.

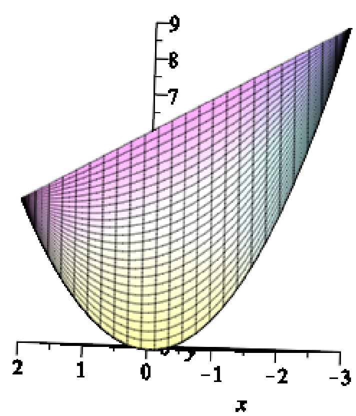
Ans: $\displaystyle\int_0^5 \int_0^4 2\sqrt{z}\,dz\,dx = \frac{160}{3}$ units^3

EXAMPLE 6: Consider the solid bounded by $z = x^2$, $x + y + z = 6$, $y = 0$.





a view down the positive z-axis



a view down the positive x-axis

1. Set up an iterated integral with order $dy\,dx$ which would compute the volume of the solid.

Ans: $\displaystyle\int_{-3}^{2}\int_{0}^{6-x-x^2} \left(6 - x - y - x^2\right)\,dy\,dx = \frac{625}{12}$ units^3

2. Set up an iterated integral with order $dz\,dx$ which would compute the volume of the solid.

Ans: $\displaystyle\int_{-3}^{2}\int_{x^2}^{6-x} (6 - x - z)\ dz\,dx = \frac{625}{12}$ units^3

HOMEWORK: Section 16.2: 9 - 89 every other odd.